Do elliptic operators have positive fundamental solutions?

Guido Sweers, Cologne University

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For a pure second order elliptic differential equation with constant coefficients the symbol $L(i\xi) = \xi^T A\xi$ uses a positive definite matrix A and the fundamental solution, defined by Fourier transformlation through

$$F(x) := \mathcal{F}^{\text{inv}}_{\xi\to x} \left(\xi^T A\xi\right)^{-1} \mathcal{F}_{x\to\xi}\delta_0(x)$$

coincides, after a transformation of coordinates, with the one for the $-\Delta$. For $n > 2$ this fundamental solution satisfies

$$F_{-\Delta}(x) = c_n |x|^{2-n}$$

and is positive. The positive singularity remains when the elliptic operator contains lower order derivatives. Also by adjusting for boundary conditions the local singularity of the Green function remains positive.

The fundamental solution for $(-\Delta)^m$, whenever $n > 2m$, satisfies

$$F_{(-\Delta)^m}(x) = c_{m,n} |x|^{2m-n}$$

and is also positive. Does the analogy continue and are fundamental solutions for pure $2m$-th order elliptic differential equations positive? The answer is yes for powers of second order operators. This is however not the case for general pure $2m$-th order elliptic differential equations.

- H.-Ch. Grunau, G. Romani, G. Sweers, Differences between fundamental solutions of general higher order elliptic operators and of products of second order operators, Mathematische Annalen 381 (2021), 1031-1084.